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Quantum Brownian Motion. II

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This paper generalizes some previous results presented in Gaioli *et al.* [*Int. J. Theor. Phys.* **36**, 2167 (1997)]. We evaluate the autocorrelation function of the stochastic acceleration and study the asymptotic evolution of the mean occupation number of a harmonic oscillator playing the role of a Brownian particle. We also analyze some deviations from the Bose population at low temperatures and compare it with the deviations from the exponential decay law of an unstable quantum system.

1. INTRODUCTION

This work is an extension of some analytical results that have already been presented in a previous paper on Brownian motion (Gaioli *et al.*, 1997), hereafter referred to as paper I.[5] In I we considered a model consisting of a Brownian oscillator of frequency Ω linearly coupled to a bath of harmonic oscillators in the rotating wave approximation. This model has an exact solution from which we studied the time evolution of the relevant physical quantities, e.g., the mean position $\langle X(t)\rangle$ and mean population $\langle N_\Omega(t)\rangle$ of the Brownian oscillator. We showed that the equation of motion governing the evolution of $\langle X\rangle$ is a generalized-local-in-time form of the Langevin equation (in mean values), with time-dependent coefficients (see also Garcia Alvarez and Gaioli, 1998). On the other hand, the equation of motion corresponding to $\langle N_\Omega\rangle$ is a generalized-local-in-time form of the master equation, with time-

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[5] Equations numbered as (#) in paper I will be labeled as (I.#) here.

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dependent coefficients [this was anticipated in I, but finally shown in Garcia Alvarez and Gaioli (1998)].

In Section 2 we reobtain the generalized Langevin equation (I.42), but in this case without considering it in mean values. Then a new term appears, which plays the role of a stochastic acceleration. Evaluating the time-dependent coefficients through a perturbative procedure up to the first relevant order, we recover the standard form of the Langevin equation, where the coefficients are independent of time and the stochastic acceleration has a zero-centered distribution leading to white noise in the classical limit.

In Section 3 we show that the same perturbative analysis mentioned above performed on equation (I.33) leads to the solution of an approximated equation for the mean population $\langle N_\Omega \rangle$ introduced by van Kampen [equation (XVII.2.30) of van Kampen (1992)]. Van Kampen's equation, which is the Born approximation of the generalized master equation derived in a previous work (Garcia Alvarez and Gaioli, 1998), makes explicit the temporal behavior of $\langle N_\Omega \rangle$: It decays exponentially until reaching thermal equilibrium with the heat bath, i.e., the Bose distribution. This behavior was shown in the figures of paper I, obtained from the exact solution of the model. However, a careful evaluation of the asymptotic value of $\langle N_\Omega \rangle$ shows that some deviations from the Bose population arise. That is, at high and intermediate temperatures the equilibrium value corresponds to the Bose distribution at the renormalized frequency $\Omega + \delta\Omega$, where $\delta\Omega$ is a shift proportional to the square of the perturbation strength, while at low temperatures a power-law behavior is found. This deviation is related to the long-time tail of the decay probability $P_{\Omega\Omega}(t)$—known as the Khalfin effect (Khalfin, 1957)—of the unstable one-particle state $|\Omega\rangle$.

In Section 4 we outline our main conclusions; there are also two appendices: Appendix A contains the details of the calculation of the autocorrelation function of the stochastic acceleration and Appendix B includes an explicit derivation of the exponential decay law and the Khalfin effect.

2. GENERALIZED LANGEVIN EQUATION AND THE "STOCHASTIC" ACCELERATION

The Hamiltonian [equation (I.5)] of the composed system is given by

$$H = \Omega\left(B^\dagger B + \frac{1}{2}\right) + \sum_{n=1}^{N} \omega_n\left(b_n^\dagger b_n + \frac{1}{2}\right) + \sum_{n=1}^{N} g_n(B b_n^\dagger + B^\dagger b_n) \quad (1)$$

The position of the subsystem Brownian oscillator $X(t) = (2M\Omega)^{-1/2}[B^\dagger(t) + B(t)]$ can be rewritten from equation (I.26) as

$$X(t) = a(t)X(0) + b(t)\frac{P(0)}{M\Omega} + f(t)$$

where

$$
\begin{aligned}
a(t) &= \sum_{\nu=0}^{N} |\Phi_\nu|^2 \cos(\alpha_\nu t) \\
b(t) &= \sum_{\nu=0}^{N} |\Phi_\nu|^2 \sin(\alpha_\nu t) \\
f(t) &= \frac{1}{\sqrt{M\Omega}} \sum_{\nu=0}^{N} \sum_{n=1}^{N} \frac{g_n}{\alpha_\nu - \omega_n} |\Phi_\nu|^2 \\
&\quad \times \left[\sqrt{m_n\omega_n} \cos(\alpha_\nu t)\, x_n(0) + \frac{\sin(\alpha_\nu t)}{\sqrt{m_n\omega_n}} p_n(0) \right]
\end{aligned}
\tag{2}
$$

and

$$|\Phi_\nu|^2 = \left[1 + \sum_{n=1}^{N} \left(\frac{g_n}{\alpha_\nu - \omega_n} \right)^2 \right]^{-1}$$

Since we have two constants of integration $X(0)$ and $P(0)$ and a particular solution $f(t)$, $X(t)$ satisfies a second-order differential equation such as

$$\ddot{X}(t) + \Omega^2(t)X(t) + \Gamma(t)\dot{X}(t) = F(t) \tag{3}$$

with the inhomogeneous term given by

$$F(t) = \ddot{f}(t) + \Omega^2(t)f(t) + \Gamma(t)\dot{f}(t) \tag{4}$$

The unknown coefficients $\Omega^2(t)$ and $\Gamma(t)$ can be easily determined by solving the linear system which results by replacing the two independent solutions of the homogeneous equation:

$$
\begin{aligned}
\ddot{a}(t) + \Omega^2(t)a(t) + \Gamma(t)\dot{a}(t) &= 0 \\
\ddot{b}(t) + \Omega^2(t)b(t) + \Gamma(t)\dot{b}(t) &= 0
\end{aligned}
$$

that is,

$$\Omega^2(t) = \frac{\dot{a}\ddot{b} - \dot{b}\ddot{a}}{a\dot{b} - b\dot{a}}, \qquad \Gamma(t) = \frac{b\ddot{a} - a\ddot{b}}{a\dot{b} - b\dot{a}} \tag{5}$$

Garcia Alvarez and Gaioli (1998) have written these coefficients in terms of the survival amplitude (and its complex conjugate) of the state $|\omega_0\rangle \equiv |\Omega\rangle = B^\dagger|0\rangle$:

$$A_{\Omega\Omega}(t) \equiv \langle\Omega|e^{-iht}|\Omega\rangle = a(t) + ib(t)$$

where

$$h = \Omega|\Omega\rangle\langle\Omega| + \sum_{n=1}^{N} \omega_n|\omega_n\rangle\langle\omega_n| + \sum_{n=1}^{N} g_n \left(|\Omega\rangle\langle\omega_n| + |\omega_n\rangle\langle\Omega|\right) + C$$

[equation (I.7); we are calling now the Hamiltonian h instead of H_1, according to the notation of Garcia Alvarez and Gaioli (1998)], and C is the zero-point energy. This survival amplitude is the cornerstone of the theory of unstable quantum systems.

On the other hand, the mean value of the "stochastic" acceleration $F(t)$ vanishes since in the thermal initial distribution (I.29) $[\rho(0) = \rho_B(0) \otimes e^{-\beta H_b}/\mathrm{tr}_b(e^{-\beta H_b})$, where $H_b = \sum_{n=1}^{N} \omega_n (b_n^\dagger b_n + 1/2)$ and tr_b is the partial trace over the reservoir], the bath operators have vanishing mean values,

$$\langle F(t)\rangle = 0 \tag{6}$$

Performing a perturbative expansion up to the first relevant order in the coupling parameter, we can recover the standard form of the Langevin equation. So, in this case we have

$$\ddot{X}(t) + (\Omega + \delta\Omega)^2 X(t) + \gamma\dot{X}(t) = F(t) \tag{7}$$

where $\delta\Omega$ is the shift of the frequency and γ the damping coefficient.

To see this we evaluate the survival amplitude up to the second order from (Sakurai, 1995)

$$A_{\Omega\Omega}(t) = e^{-i\Omega t} c_{\Omega\Omega} \tag{8}$$

where

$$c_{\Omega\Omega} = 1 - \sum_{k=1}^{N} g_k^2 \int_0^t \int_0^{t'} e^{i\omega_{\Omega k}t'} e^{i\omega_k\Omega t''} dt'\, dt'' \tag{9}$$

and $\omega_{\Omega k} = \Omega - \omega_k$. Taking the derivative of equation (9) with respect to time, up to second order, we obtain

$$\frac{\dot{c}_{\Omega\Omega}}{c_{\Omega\Omega}} = 1 - i \sum_{k\neq 0}^{N} g_k^2 \left(-i \int_0^t e^{i\omega_{\Omega k}\tau} d\tau\right)$$

Performing a long-time approximation ($t >> 1/\Omega$), we obtain

$$\lim_{t\to\infty} \left(-i \int_0^t e^{i\alpha\tau}\, d\tau\right) = \delta_+(\alpha) = \frac{1}{\alpha + i\varepsilon} = \mathrm{Pv}\,\frac{1}{\alpha} - i\pi\delta(\alpha) \tag{10}$$

where Pv denotes the principal value. Therefore the survival amplitude of the state $|\Omega\rangle$ has a decaying exponential contribution [cf. equation (B10)]

$$A_{\Omega\Omega}(t) = e^{-i(\Omega+\delta\Omega-i\gamma/2)t} \tag{11}$$

where[6]

$$\delta\Omega = \mathrm{Pv} \sum_{k\neq 0}^{N} \frac{g_k^2}{\Omega - \omega_k} \tag{12}$$

and

$$\gamma = 2\pi \sum_{k\neq 0}^{N} g_k^2 \delta(\omega_k - \Omega) \tag{13}$$

If we now use the second-order value (11) of the amplitude $A_{\Omega\Omega}$ in equation (5), a straightforward evaluation leads to

$$\Omega(t) = \Omega + \delta\Omega, \qquad \Gamma(t) = \gamma \tag{14}$$

so we retrieve standard expressions usually derived from the Born and Markovian approximations. In our case the Markovian approximation is not necessary because equation (3) is just local in time, in contrast with the better known integrodifferential form commonly found in the literature (Louisell, 1973, 1977; Sargent *et al.*, 1974; Lindenberg and West, 1984, 1990; Ford *et al.*, 1988a, b; Meystre and Sargent, 1991; Cohen-Tannoudji *et al.*, 1992; Mandel and Wolf, 1995).

The autocorrelation function of the "stochastic" acceleration is defined by

$$K(t) = \frac{1}{2} \langle F(0)F(t) + F(t)F(0)\rangle$$

Up to the second order, by means of a long, but straightforward calculation (see Appendix A), we obtain

$$K(t) = \frac{1}{2M\Omega} \sum_{m=1}^{N} \{[2\langle N_m(0)\rangle + 1] g_m^2 (\omega_m + \Omega)^2 \cos \omega_m t\} \tag{15}$$

For an initial thermal distribution for the bath oscillators, i.e., $2\langle N_m(0)\rangle + 1 = \coth(\beta\omega_m/2)$, and by taking the limit of a continuous bath (see paper I), $K(t)$ becomes

$$K(t) = \frac{1}{2M} \int_0^\infty d\omega\, g^2(\omega) \frac{(\omega + \Omega)^2}{\Omega} \coth \frac{\beta\omega}{2} \cos \omega t$$

where $g^2(\omega)$ is defined as

[6] The following expressions must be understood as if a continuous limit has been taken in such a way that summations become integrals. On the contrary, we must replace the principal part and delta distributions by their corresponding approximants.

$$g^2(\omega)\Delta\omega = \sum_{n|(\omega<\omega_n<\omega+\Delta\omega)} g_n^2$$

(Ullersma, 1966; van Kampen, 1992).

Considering a coupling function $g^2(\omega)$ which is peaked[7] around $\omega = \Omega$, then we can replace $\frac{1}{2}g^2(\omega)(\omega + \Omega)^2/\Omega$ by $(\omega/\pi)2\pi g^2(\Omega)$ in the integrand. Taking into account that, up to the second order [see equation (13)], the damping factor in the Langevin equation is $\gamma = 2\pi g^2(\Omega)$, we finally have

$$K(t) \approx \frac{\gamma}{M\pi}\int_0^\infty d\omega\, \hbar \coth\frac{\beta\hbar\omega}{2}\cos\omega t = \frac{\gamma k_B T}{M}\frac{d}{dt}\coth\left(\frac{\pi k_B T t}{\hbar}\right)$$

In the classical limit ($\hbar \to 0$) it goes to the classical stochastic, delta-correlated, autocorrelation function

$$\lim_{\hbar\to 0} K(t) = \frac{2\gamma k_B T}{M}\,\delta(t) \tag{16}$$

originally proposed by Langevin.

Equation (16) corresponds to instantaneous correlated fluctuations, which leads to a Markovian process (instantaneous memory loss, since the values of the stochastic acceleration at two different times are not correlated). $F(t)$ is the source of noise (fluctuations) known as *white noise*. In this limit the usual Langevin equation (7) together with properties (6) and (16) are recovered from a general formulation. A new ingredient is that in equation (7) the oscillator frequency is shifted by a value $\delta\Omega$.

3. BEHAVIOR OF THE MEAN POPULATION OF THE BROWNIAN OSCILLATOR

Van Kampen (1992) showed that the elimination of fast microscopic variables leads to a closed expression for the mean value of the occupation number of the Brownian oscillator

$$\frac{d}{dt}\langle N_\Omega(t)\rangle = -\gamma\langle N_\Omega(t)\rangle + \frac{\gamma}{e^{\beta\Omega} - 1} \tag{17}$$

where irrelevant variables get in only through the initial thermal state at a temperature $T = 1/k_B\beta$.

In what follows we use the time-dependent perturbation theory to derive equation (17) starting from equation (I.33) of I:

[7] This condition is necessary in order for the rotating wave approximation to be valid (see, e.g., Gaioli, 1997).

$$\langle N_\Omega(t)\rangle = P_{\Omega\Omega}(t)\,\langle N_\Omega(0)\rangle + \sum_{n=1}^{N} P_{\Omega n}(t)\langle N_n(0)\rangle \tag{18}$$

We consider that the perturbation [interaction of equation (1)] is time independent and turns on at $t = 0$. In this case the Dyson series for the transition amplitude to second order can be written as (Sakurai, 1995)

$$A_{nm}(t) = e^{-i\omega_m t}\left(c_{mn}^{(0)} + c_{mn}^{(1)} + c_{mn}^{(2)} + \dots\right) \tag{19}$$

where

$$\begin{aligned} c_{mn}^{(0)} &= \delta_{mn} \\ c_{mn}^{(1)} &= -i\int_0^t e^{i\omega_{mn}t'}\, v_{mn}\, dt' \\ c_{mn}^{(2)} &= -\sum_{k=0}^{N}\int_0^t\int_0^{t'} e^{i\omega_{mk}t'}\, v_{mk} e^{i\omega_{kn}t''}\, v_{kn}\, dt'\, dt'' \end{aligned} \tag{20}$$

with $v_{mn} = \langle\omega_m|v|\omega_n\rangle = g_m\delta_{n,0} + g_n\delta_{m,0}$, except for $v_{00} = 0$, and $\omega_{mn} = \omega_m - \omega_n$. The evaluation of the transition probabilities to second order gives

$$P_{nm} = \delta_{nm} + \Gamma_{nm}t \tag{21}$$

with

$$\Gamma_{nm} = 2\pi v_{nm}^2\, \delta_t(\omega_n - \omega_m) \qquad \text{for} \quad n \neq m \tag{22}$$

and

$$\Gamma_{nm} = -2\pi \sum_{m\neq n} v_{nm}^2\, \delta_t(\omega_n - \omega_m) \tag{23}$$

where $\delta_t(\alpha) \equiv (\sin^2 \alpha t/\pi\alpha^2 t)$ is a function which approaches Dirac's delta when time goes to infinity.[8] Introducing these results into equation (18) and taking into account that $\Gamma_{00} = \Gamma_{\Omega\Omega} \equiv -\gamma$ and that, in this case, $P_{\Omega\Omega} \simeq 1 - \gamma t$, we obtain

[8] Let η be the width of the interaction function g_n^2. Since the function $\delta_t\,(\alpha)$ has a width $4\pi/t$, in order to behave as a delta distribution we need times such that $t >> 4\pi/\eta$. This is the meaning of long times. We also see that the times involved in this approximation must satisfy $t << 1/\gamma$, where $\gamma = -\Gamma_{00}$, in order for equation (21) to be valid. This condition and the long-time limit restrict the time range to $4\pi/\eta << t << 1/\gamma$.

$$\langle N_\Omega\,(t)\rangle \simeq (1-\gamma t)\,\langle N_\Omega(0)\rangle + \gamma t\langle N_{\omega_n=\Omega}(0)\rangle$$

$$= (1-\gamma t)\,\langle N_\Omega(0)\rangle + \frac{\gamma t}{e^{\beta\Omega}-1}$$

which is the second-order perturbative expansion of

$$\langle N_\Omega(t)\rangle \simeq e^{-\gamma t}\,\langle N_\Omega(0)\rangle + (1-e^{-\gamma t})\,\frac{1}{e^{\beta\Omega}-1} \tag{24}$$

Equation (24) corresponds to the solution of equation (17) [cf. equation (XVII.2.29) of van Kampen (1992)]. This is also the second-order approximation of the solution of the exact master equation of Garcia Alvarez and Gaioli (1998).

We emphasize that this is a perturbative result. A full exact calculation can be carried out in the limit of a continuous bath (as it was taken in Section 6 of paper I). We now calculate the asymptotic long-time equilibrium value of the mean population of the Brownian particle under this limit. We begin with equation (I.76), but setting ω_{min} and ω_{max} equal to zero and infinity, respectively, so we have

$$\langle N_\Omega\,(\infty)\rangle = \int_0^\infty d\omega\,\frac{g^2(\omega)}{|R_+^{-1}(\omega)|^2}\,\frac{1}{e^{\beta\omega}-1} \tag{25}$$

By considering now "no-low" temperatures and a small coupling, it is easy to see that $g^2(\omega)\,|R_+^{-1}\,(\omega)|^{-2}$ picks the value at $\omega = \Omega + \delta\Omega$ and therefore has a $\delta(\omega - \Omega - \delta\Omega)$ behavior.
Thus equation (25) leads to the thermal equilibrium value at the shifted frequency $\Omega + \delta\Omega$, i.e.,

$$\langle N_\Omega(\infty)\rangle = \langle N_{\omega=\Omega+\delta\Omega}(0)\rangle = \frac{1}{e^{\beta(\Omega+\delta\Omega)}-1} \tag{26}$$

In the classical limit $\langle N_\Omega(\infty)\rangle \approx [\beta(\Omega+\delta\Omega)]^{-1}$, which leads to the equipartition of energy $E \approx k_B T$ (two quadratic degrees of freedom in a one-dimensional configuration space) and the heat capacity $C_v \approx k_B$.

At low temperatures, deviations (an inverse power-law falloff in expectation values) from this equilibrium distribution were already reported by some authors (Lindenberg and West, 1984, 1990; Haake and Reibold, 1985; Joichi *et al.*, 1997), since the effect of the coupling becomes macroscopically observable. This effect is similar to the deviation from the exponential decay law for long times first described by Khalfin (1957). In Appendix B we explicitly calculate the survival amplitude using the resolvent method. In such a case we show that $A_{\Omega\Omega}$ reduces to [equation (B8)]

$$A_{\Omega\Omega}(t) = \int_0^\infty d\omega \frac{g^2(\omega)}{|R_+^{-1}(\omega)|^2} e^{-i\omega t} \tag{27}$$

Comparing equation (27) with the low-temperature limit of equation (25),

$$\langle N_\Omega(\infty)\rangle = \int_0^\infty d\omega \frac{g^2(\omega)}{|R_+^{-1}(\omega)|^2} e^{-\beta\omega} \tag{28}$$

we can see that, by making the identification $\beta = it$, both expressions are equivalent. That is, the low-temperature regime of the mean occupation number corresponds to the long-time behavior of the survival amplitude of the unstable state $|\Omega\rangle$. This is one of the deep interrelationships between statistical properties of nonequilibrium ensembles and the behavior of unstable quantum systems. This was possible since our model [equation (1)] can be decomposed by sectors of a fixed number of quanta [see equation (I.6)], which is one of the reasons for our choice of the model.

We then use the long-time limit of the survival amplitude (Khalfin effect) of Appendix B [equation (B12)] in order to estimate the low-temperature anomalous behavior of the mean population, namely

$$\langle N_\Omega(\infty)\rangle \sim (k_B T)^{n+1} \tag{29}$$

The energy of the Brownian oscillator behaves like $E \sim \mathrm{T}^{n+1}$ and so $C_V \sim T^n$, which is an analogous result to the low-temperature behavior of the Debye model of phonons (Huang, 1987). One of the advantages of this kind of calculation is that equation (29) can be experimentally measured and then it provides an indirect proof of the existence of the Khalfin effect, which is very difficult to measure because of the time scale involved. However, the other deviation from the exponential decay law, the Zeno effect (Misra and Sudarshan, 1977), was recently measured for the first time (Wilkinson *et al.*, 1997).

We have seen that the anomalous behavior of the mean population at low temperatures [equation (29)] is related to the deviations of the exponential decay law of the unstable initially prepared state $|\Omega\rangle$ at very long times. We can also see another relation between this anomaly and a generalized statistics proposed by Tsallis (1988) 10 years ago (see also Curado and Tsallis, 1991). Let us see the origin of this conjecture.

Büyükkiliç and Demirhan (1993) and Büyükkiliç *et al.* (1995) have shown that for a set of bosons, labeled by the index k, the mean occupation number corresponding to the generalized Bose–Einstein canonical distribution is approximately[9] given by

[9] The fact that this is a nonexact result was noticed by Pennini *et al.* (1995).

$$\langle N_k \rangle \simeq \frac{1}{[1 + (q - 1)\beta\omega_k]^{1/(q-1)} - 1} \tag{30}$$

where q is a parameter which characterizes the nonextensive nature of the system, and thus depends on the long-range nature of interactions present in the system. This parameter is such that for $q \to 1$ one retrieves the standard results, i.e., the Bose–Einstein mean population. For $q \neq 1$, in the low-temperature regime, we can neglect the terms independent of temperature in the denominator of expression (30). Then, we have

$$\langle N_k \rangle \sim (k_B T)^{1/(q-1)} \tag{31}$$

Compare equation (31) with equation (29). We see that the equilibrium distribution reached by the Brownian oscillator resembles that of the bosons in thermal equilibrium for a nonextensive system, according to Tsallis' prescription. From these equations we obtain

$$q = \frac{n + 2}{n + 1}$$

a number which satisfies $1 < q < 2$, since $n > 0$. Maybe the explanation of this behavior is that the Brownian oscillator is not able to cover all accessible quantum states (as a consequence of strong quantum correlations at low temperatures) and then it cannot reach the most probable distribution according to the ergodic hypothesis.

4. CONCLUSIONS

In this paper the autocorrelation function of the stochastic acceleration and the asymptotic mean population of the Brownian oscillator were analytically evaluated from a deterministic quantum dynamics. As regards the Langevin equation, we have provided the stochastic term which was skipped in paper I. For the mean occupation number we have found that it reaches thermal equilibrium at the bath temperature corresponding to the Bose population. At low temperatures a deviation from this population was found which has a common origin with the deviations from the exponential decay law. However, the Khalfin effect is very difficult to measure since the usual observation times of unstable quantum systems are much shorter than the time the decay law is no longer exponential.

APPENDIX A. AUTOCORRELATION FUNCTION OF THE STOCHASTIC ACCELERATION

According to equation (4), $K(t)$ is given by

$$
\begin{aligned}
K(t) = \frac{1}{2}[\langle & \ddot{f}(0)\ddot{f}(t) + \ddot{f}(0)\Omega^2(t)f(t) + \ddot{f}(0)\Gamma(t)\dot{f}(t) \\
& +\Omega^2(0)f(0)\ddot{f}(t) + \Omega^2(0)f(0)\Omega^2(t)f(t) + \Omega^2(0)f(0)\Gamma(t)\dot{f}(t) \\
& +\Gamma(0)\dot{f}(0)\ddot{f}(t) + \Gamma(0)\dot{f}(0)\Omega^2(t)f(t) + \Gamma(0)\dot{f}(0)\Gamma(t)\dot{f}(t)\rangle \\
& + \langle 0 \leftrightarrow t\rangle] \qquad \text{(A1)}
\end{aligned}
$$

where $\langle 0 \leftrightarrow t\rangle$ stands for interchanging $t = 0$ with t. $f(t)$ can be written in terms of the $A_{\Omega m}$ as

$$
f(t) = \frac{1}{\sqrt{2M\Omega}} \sum_{m=1}^{N} [A_{\Omega m}(t)b_m^{\dagger}(0) + h.c.] \qquad \text{(A2)}
$$

Considering second-order contributions only and taking into account that $\delta\Omega$ and Γ are time independent up to this order, we can rewrite equation (A1) as

$$
\begin{aligned}
K(t) = \frac{1}{2}[\langle & \ddot{f}(0)\ddot{f}(t) + \Omega^2\ddot{f}(0)f(t) + \Omega^2 f(0)\ddot{f}(t) + \Omega^4 f(0)f(t)\rangle \\
& +\langle 0 \leftrightarrow t\rangle]
\end{aligned}
$$

since the f's are linear in the $A_{\Omega m}$ [see equation (A2)]. We need to expand the amplitudes up to the first order. From equations (19) and (20) we have

$$
A_{\Omega m}(t) = -ie^{-\omega_m t} v_{m\Omega} \int_0^t e^{i(\omega_m - \Omega)t'} dt'
$$

Solving the integral, we have

$$
A_{\Omega m}(t) = \frac{v_{m\Omega}}{\omega_m - \Omega}(e^{-i\omega_m t} - e^{-i\Omega t})
$$

The second derivative of $A_{\Omega m}(t)$ appearing in $\ddot{f}$ is given by

$$
\ddot{A}_{\Omega m}(t) = \frac{v_{m\Omega}}{\omega_m - \Omega}(-\omega_m^2 e^{-i\omega_m t} + \Omega^2 e^{-i\Omega t})
$$

Taking into account that $A_{\Omega m}(0) = 0$, from which $f(0) = 0$, then we must only calculate

$$K(t) = \frac{1}{2}\,[\langle \ddot{f}(0)\ddot{f}(t) + \Omega^2 \ddot{f}(0) f(t)\rangle + \langle 0 \leftrightarrow t\rangle]$$

Let us see each term step by step. Let K_1 and K_2 be defined by

$$K_1(t) = \frac{1}{2}\,\langle\, \ddot{f}(0)\ddot{f}(t) + \ddot{f}\,(t)\ddot{f}(0)\rangle$$

and

$$K_2(t) = \frac{\Omega^2}{2}\,\langle \ddot{f}(0) f(t) + f(t)\,\ddot{f}(0)\rangle$$

Therefore

$$K_1(t) = \frac{1}{2}\frac{1}{M\Omega}\sum_{m,m'=1}^{N}[\ddot{A}_{\Omega m}(0)\ddot{A}^*_{\Omega m'}(t)\,\langle b^\dagger_m b_{m'}\rangle(0) + \ddot{A}^*_{\Omega m}(0)\ddot{A}_{\Omega m'}(t)\langle b_m b^\dagger_{m'}\rangle(0)$$
$$+\ddot{A}_{\Omega m}(t)\ddot{A}^*_{\Omega m'}(0)\,\langle b^\dagger_m b_{m'}\rangle(0) + \ddot{A}^*_{\Omega m}(t)\,\ddot{A}_{\Omega m'}(0)\,\langle b_m b^\dagger_{m'}\rangle(0)]$$

which, from equation (I.30) and the commutation relations (I.4), is reduced to

$$K_1(t) = \frac{1}{4M\Omega}\sum_{m=1}^{N}\{\mathrm{Re}[\ddot{A}_{\Omega m}(0)\ddot{A}^*_{\Omega m}(t)]\,[2\langle N_m\rangle(0)+1]\}$$
$$= \frac{1}{2M\Omega}\sum_{m=1}^{N}\left\{[2\langle N_m\rangle(0)+1]\frac{|v_{m\Omega}|^2}{\omega_m-\Omega}(\omega_m+\Omega)\,(\omega_m^2\cos\omega_m t - \Omega^2\cos\Omega t)\right\}$$

An analogous calculation for $K_2(t)$ leads to

$$K_2(t) = \frac{\Omega}{2M}\sum_{m=1}^{N}\{\mathrm{Re}[\ddot{A}_{\Omega m}(0)\ddot{A}^*_{\Omega m}(t)]\,[2\langle N_m\rangle(0)+1]\}$$
$$= \frac{1}{2M\Omega}\sum_{m=1}^{N}\left\{[2\langle N_m\rangle(0)+1]\frac{|v_{m\Omega}|^2}{\omega_m-\Omega}(\omega_m+\Omega)\,(\Omega^2\cos\Omega t - \Omega^2\cos\omega_m t)\right\}$$

Joining K_1 and K_2, we finally obtain equation (15)

$$K(t) = \frac{1}{2M\Omega}\sum_{m=1}^{N}\{[2\langle N_m\rangle(0) + 1]|v_{m\Omega}|^2(\omega_m+\Omega)^2\cos\omega_m t$$

APPENDIX B. EVALUATION OF THE SURVIVAL AND TRANSITION AMPLITUDES

We analyze the analytical structure of $A_{\Omega\Omega}(t)$ in order to show how an exponential contribution arises for a significant range of time and how deviations from this behavior appear. Using the well-known identity between distributions

$$\frac{1}{x \pm i\epsilon} = \text{PV}\,\frac{1}{x} \mp i\pi\delta(x) \tag{B1}$$

we can obtain, for $x = \omega - h$, the following integral representation of the evolution operator:

$$e^{-iht} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,[G_-(\omega) - G_+(\omega)] \tag{B2}$$

where $G_\pm(\alpha) = 1/(\alpha \pm i\epsilon - h)$ is the retarded $(+)$ [advanced $(-)$] Green function of the time-independent Schrödinger equation (the resolvent), corresponding to the total Hamiltonian. From (B2) we can evaluate the survival and transition amplitudes. Such a calculation involves the knowledge of the partial resolvents (Schwinger, 1961; Messiah, 1962), departing from

$$(\alpha \pm i\epsilon - h)\, G_\pm(\alpha) = I \tag{B3}$$

By taking the matrix elements in equation (B3), we have

$$(\alpha \pm i\epsilon - \Omega)\,\langle\Omega|\, G_\pm(\alpha)|\Omega\rangle + \int_0^{\infty} d\omega\, g(\omega)\langle\omega|G_\pm(\alpha)|\,\Omega\rangle = 1 \tag{B4}$$

$$(\alpha \pm i\epsilon - \omega)\langle\omega|G_\pm(\alpha)|\Omega\rangle = g(\omega)\langle\Omega|G_\pm(\alpha)|\Omega\rangle = 0 \tag{B5}$$

So, from (B4) and (B5), the desired matrix elements of the resolvent are

$$\langle\Omega|G_\pm(\alpha)|\Omega\rangle = \frac{1}{\alpha \pm i\epsilon - \Omega - \int_0^{\infty} d\,\omega\,[g^2(\omega)/(\alpha \pm i\epsilon - \omega)]} = R_\pm(\alpha)$$

and

$$\langle\omega|G_\pm(\alpha)|\Omega\rangle = \frac{g(\omega)}{\alpha \pm i\epsilon - \omega}\,\langle\Omega|G_\pm(\alpha)|\Omega\rangle$$

Now returning to (B2), we can obtain the survival and transition amplitudes as

$$\begin{aligned} A_{\Omega\Omega}(t) &= \langle\Omega|e^{-iht}|\Omega\rangle \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega'\, e^{-i\omega' t}\,[R_-(\omega') - R_+(\omega')] \end{aligned} \tag{B6}$$

$$A_{\Omega\omega}(t) = \langle\omega|e^{-iht}|\Omega\rangle$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega'\, e^{-i\omega' t}\left[\frac{g(\omega')R_-(\omega')}{\omega - i\epsilon - \omega'} - \frac{g(\omega')R_+(\omega')}{\omega + i\epsilon - \omega'}\right] \tag{B7}$$

i.e., taking the Fourier transform of the difference between the advanced and retarded reduced resolvents (note that we have not introduced any initial condition for the amplitudes). It can be proved that the reduced resolvents have the general expression

$$R_\pm(\omega) = \frac{1}{\omega \pm i\epsilon - \Omega - \Sigma_\pm(\omega)}$$

where $\Sigma(\omega)$ is the level-shift operator in the subspace generated by $|\Omega\rangle$. The reduced resolvent $R_+(\omega)$ [$R_-(\omega)$] is the analog of the exact Feynman (Dyson) electron propagator with $\Sigma_\pm(\omega)$ playing the role of the Dyson (1949) mass operator. In our case $\Sigma_\pm(\alpha) = \int_0^\infty d\omega\, [g^2(\omega)/(\alpha \pm i\epsilon - \omega)]$. Using (B1), we can rewrite it as $\Sigma_\pm(\alpha) = \Delta(\alpha) \mp i\gamma(\alpha)/2$, with

$$\Delta(\alpha) = \mathrm{PV}\int_0^\infty d\omega\, \frac{g^2(\omega)}{\alpha - \omega}$$
$$\gamma(\alpha) = 2\pi g^2(\alpha)$$

which are nothing else than equations (12) and (13) in the case of a continuous bath. Taking into account that $g(\omega) = 0$ for $\omega < 0$, we can rewrite equation (B6) as

$$A_{\Omega\Omega}(t) = \int_0^\infty d\omega\, \frac{g^2(\omega)}{|R_\mp^{-1}(\omega)|^2}\, e^{-i\omega t} \tag{B8}$$

In the theory of unstable states (Messiah, 1962; Goldberger and Watson, 1964; Cohen-Tannoudji *et al.*, 1992) it is common to find $A_{\Omega\Omega}$ written as

$$A_{\Omega\Omega}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{\gamma(\omega)}{[\omega - \Omega - \Delta(\omega)]^2 + \frac{1}{4}\gamma^2(\omega)}\, e^{-i\omega t} \tag{B9}$$

which allows one to easily study the different decay regimes. If $\gamma(\omega)$ is small, the term $[\omega - \Omega - \Delta(\omega)]^2$ is large compared with $\gamma^2(\omega)$ except when $\omega \simeq \Omega + \Delta(\Omega)$. Thus we replace $\gamma(\omega)$ by $\gamma(\Omega) \equiv \gamma$ and $\Delta(\omega)$ by $\Delta(\Omega) \equiv \delta\Omega$. The Lorentzian function resulting from this replacement is known as the

Breit–Wigner (1936) distribution. In this case equation (B9) has an analytical result

$$A_{\Omega\Omega}(t) = e^{-i(\Omega+\delta\Omega)t}\, e^{-\delta t/2} \tag{B10}$$

which, as expected, retrieves the well-known exponential decay law as originally derived by Weisskopf and Wigner (1930). Some deviations from this exponential decay arise as we inspect equation (B9) more carefully. If we retain $\gamma(\omega)$ in the numerator of (B9), we can rewrite this equation as

$$A_{\Omega\Omega}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{\gamma}{[\omega - \Omega - \delta\Omega]^2 + \frac{1}{4}\gamma^2} \frac{\gamma(\omega)}{\gamma}\, e^{-i\omega t} \tag{B11}$$

The r.h.s. of equation (B11) is the convolution product of (B10) and the Fourier transform of $\gamma^{-1}\gamma(\omega)$. Since $\gamma(\omega)$ has a finite width, its Fourier transform $\gamma(t)$ has also a finite width. We also have that $\gamma(\omega)$ is null for $\omega \leq 0$ and is not infinitely differentiable at $\omega = 0$. Then, if we suppose that $\gamma(\omega)$ goes as ω^n for small ω, then $\gamma(t)$ behaves like $t^{-(n+1)}$ for very long times. This is known as the Khalfin (1957) effect, namely

$$A_{\Omega\Omega}(t) \sim 1/t^{n+1} \tag{B12}$$

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